CAUCHY-SCHWARZ-TYPE INEQUALITIES ON KÄHLER MANIFOLDS-II

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ABSTRACT. We establish in this note some Cauchy-Schwarz-type inequalities on compact Kähler manifolds, which generalize the classical Khovanskii-Teissier inequalities to higher-dimensional cases. Our proof is to make full use of the mixed Hodge-Riemann bilinear relations due to Dinh and Nguyên. A proportionality problem related to our main result is also proposed.

1. Introduction and main results

Suppose X is an n-dimensional algebraic manifold and $D_1, D_2, \ldots, D_n$ are n (not necessarily distinct) ample divisors on X. Then we have the following opposite Cauchy-Schwarz-type inequality,

$$([D_1 D_2 D_3 \cdots D_n])^2 \geq [D_1 D_1 D_3 \cdots D_n] \cdot [D_2 D_2 D_3 \cdots D_n], \tag{1.1}$$

where $[\cdot]$ denotes the intersection number of the divisors inside it and the equality holds if and only if the two divisors D_1 and D_2 are numerically proportional.

(1.1) was discovered independently by Khovanskii and Teissier around in 1979 ([9],[11]) and now is called Khovanskii-Teissier inequality. This equality is indeed a generalization of the classical Aleksandrov-Fenchel inequalities and thus present a nice relationship between the theory of mixed volumes and algebraic geometry ([6, p. 114]). The proof of (1.1) is to apply the usual Hodge-Riemann bilinear relations ([7, p. 122-123]) to the Kähler classes determined by these divisors and an induction argument. The approach also suggests that the usual Hodge-Riemann bilinear relations may be extended to the mixed case. After some partial results towards this direction ([8],[13]), this aim was achieved in its full generality by Dinh and Nguyên in [2].

We would like to point out a fact, which was not mentioned explicitly in [2], that (1.1) can now be extended by the mixed Hodge-Riemann bilinear relations as follows. Suppose $\omega, \omega_1, \omega_2, \ldots, \omega_{n-2}$ are $n-1$ Kähler classes and $\alpha \in H^{1,1}(M, \mathbb{R})$ an *arbitrary* real-valued $(1,1)$-form on an n-dimensional compact connected Kähler manifold M. Then we have

$$\Big(\int_M \alpha \wedge \omega \wedge \omega_1 \wedge \cdots \wedge \omega_{n-2}\Big)^2 \geq \Big(\int_M \alpha^2 \wedge \omega_1 \wedge \cdots \wedge \omega_{n-2}\Big) \cdot \Big(\int_M \omega^2 \wedge \omega_1 \wedge \cdots \wedge \omega_{n-2}\Big), \tag{1.2}$$

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where the equality holds if and only if $\alpha \in \mathbb{R}\omega$. Indeed, [2, Theorem A] tells us that the index of the following bilinear form

$$Q(u,v) := \int_M u \wedge v \wedge \omega_1 \wedge \cdots \wedge \omega_{n-2}, \qquad u, v \in H^{1,1}(M,\mathbb{R}), \tag{1.3}$$

is of the form $(+,-,\cdots,-)$, i.e., the positive and negative indices are 1 and $h^{1,1}-1$ respectively, where $h^{1,1}$ is the corresponding Hodge number of M $\big($the dimension of $H^{1,1}(M,\mathbb{R})\big)$. Define a real-valued function $f(t) := Q(\omega+t\alpha, \omega+t\alpha)$ $(t \in \mathbb{R})$. Then $f(0) > 0$ as $\omega, \omega_1, \cdots, \omega_{n-2}$ are all Kähler classes and thus their product is strictly positive. $\omega + t\alpha$ spans a 2-dimensional subspace in $H^{1,1}(M,\mathbb{R})$ if α and ω are linearly independent and thus $f(t_0) < 0$ for some $t_0 \in \mathbb{R}$ in view of the index of $Q(\cdot,\cdot)$. Then the discriminant of $f(t)$ gives (1.2) with strict sign " $>$ ".

When these Kähler classes are all equal: $\omega = \omega_1 = \cdots = \omega_{n-2}$, (1.2) degenerates to the following special case:

$$\Big(\int_M \alpha \wedge \omega^{n-1}\Big)^2 \geq \Big(\int_M \alpha^2 \wedge \omega^{n-2}\Big) \cdot \Big(\int_M \omega^n\Big), \qquad \forall\ \alpha \in H^{1,1}(M,\mathbb{R}), \tag{1.4}$$

which is quite well-known and, to the author's best knowledge, should be due to Apte in [1]. Inspired by (1.4), the author asked in [10] whether or not there exists a similar inequality to (1.4) for those $\alpha \in H^{p,p}(M,\mathbb{R})$ $(1 \leq p \leq [\frac{n}{2}])$ and obtained a related result ([10, Theorem 1.3]), whose proof is also based on the usual Hodge-Riemann bilinear relations. As an application we presented some Chern number inequalities when the Hodge numbers of the manifolds satisfy some constraints ([10, Corollary 1.5]). Now keeping the mixed Hodge-Riemann bilinear relations established in [2] in mind, we may also ask if the main idea of the proof in [10] can be carried over to the mixed case to extend the α in (1.2) to $H^{p,p}(M,\mathbb{R})$ for $1 \leq p \leq [\frac{n}{2}]$. The answer is affirmative and this is the main goal of our current article. So this article can be viewed as a sequel to [10], which explains its title either.

Our main result (Theorem 1.3) will be stated in the rest of this section. In Section 2 we briefly review the mixed Hodge-Riemann bilinear relations and then present the proof of Theorem 1.3. In Section 3 we discuss a proportionality problem related to (1.1) posed by Teissier and propose a similar problem related to our main result.

In order to state our result as general as possible, we would like to investigate the elements in $H^{p,p}(M,\mathbb{C})$, i.e., complex-valued (p,p)-forms on M. The following definition is inspired by (1.2) and is a mixed analogue to [10, Definition 1.1].

Definition 1.1. Suppose M is an n-dimensional compact connected Kähler manifold. For $1 \leq p \leq [\frac{n}{2}]$, $\alpha \in H^{p,p}(M,\mathbb{C})$ and $n-2p+1$ Kähler calsses $\omega, \omega_1, \ldots, \omega_{n-2p}$, we put $\Omega_p := \omega_1 \wedge \cdots \wedge \omega_{n-2p}$ and define

$$g(\alpha,\omega;\Omega_p) := \Big(\int_M \alpha \wedge \bar{\alpha} \wedge \Omega_p\Big) \cdot \Big(\int_M \omega^{2p} \wedge \Omega_p\Big) - \Big(\int_M \alpha \wedge \omega^p \wedge \Omega_p\Big) \cdot \Big(\int_M \bar{\alpha} \wedge \omega^p \wedge \Omega_p\Big).$$

α is said to satisfy Cauchy-Schwarz (resp. opposite Cauchy-Schwarz) inequality with respect to the Kähler classes ω and $(\omega_1, \ldots, \omega_{n-2p})$ if $g(\alpha,\omega;\Omega_p) \geq 0$ (resp. $g(\alpha,\omega;\Omega_p) \leq 0$).

Remark 1.2. Note that $\alpha \in H^{p,p}(M,\mathbb{R})$ if and only if $\alpha = \bar{\alpha}$. Also note that $g(\alpha,\omega;\Omega_p)$ in the above definition is a real number and so we can discuss its non-negativity or non-positivity.

The main result of this note, which extends [10, Theorem 1.3] to the mixed case, is the following

Theorem 1.3. *Suppose M is an n-dimensional compact connected Kähler manifold.*

(1) *Given $1 \leq p \leq [\frac{n}{2}]$, all elements in $H^{p,p}(M,\mathbb{C})$ satisfy Cauchy-Schwarz inequality with respect to any Kähler classes ω and $(\omega_1,\dots,\omega_{n-2p})$ (in the sense of Definition 1.1) if and only if the Hodge numbers of M satisfy*

$$h^{2i,2i} = h^{2i+1,2i+1}, \qquad 0 \leq i \leq [\frac{p+1}{2}] - 1. \tag{1.5}$$

(2) *All elements in $H^{1,1}(M,\mathbb{C})$ satisfy opposite Cauchy-Schwarz inequality with respect to any Kähler classes ω and $(\omega_1,\dots,\omega_{n-2p})$.*

(3) *Given $2 \leq p \leq [\frac{n}{2}]$, all elements in $H^{p,p}(M,\mathbb{C})$ satisfy opposite Cauchy-Schwarz inequality with respect to any Kähler classes ω and $(\omega_1,\dots,\omega_{n-2p})$ if and only if the Hodge numbers of M satisfy*

$$h^{2i-1,2i-1} = h^{2i,2i}, \qquad 1 \leq i \leq [\frac{p}{2}]. \tag{1.6}$$

Moreover, in all the cases mentioned above, the equalities hold if and only if these α are proportional to ω^p.

The first part of the following corollary extends the Khovanskii-Teissier inequalities (1.1) and (1.2).

Corollary 1.4.

(1)

$$\begin{aligned}&\Big(\int_M \alpha\wedge\omega\wedge\omega_1\wedge\cdots\wedge\omega_{n-2}\Big)\cdot\Big(\int_M \bar{\alpha}\wedge\omega\wedge\omega_1\wedge\cdots\wedge\omega_{n-2}\Big)\\ \geq&\Big(\int_M \alpha\wedge\bar{\alpha}\wedge\omega_1\wedge\cdots\wedge\omega_{n-2}\Big)\cdot\Big(\int_M \omega^2\wedge\omega_1\wedge\cdots\wedge\omega_{n-2}\Big)\end{aligned}$$

for any Kähler classes $\omega,\omega_1,\dots,\omega_{n-2}$ and any $\alpha \in H^{1,1}(M,\mathbb{C})$, where the equality holds if and only if $\alpha \in \mathbb{C}\omega$.

(2) If $h^{1,1} = 1$, then all elements in $H^{2,2}(M,\mathbb{C})$ $(n \geq 4)$ satisfy Cauchy-Schwarz inequality with respect to any Kähler classes in the sense of Definition 1.1. Moreover, the equality case holds if and only if the element is proportional to ω^2.

(3) If $h^{1,1} = h^{2,2}$, then all elements in $H^{2,2}(M,\mathbb{C})$ $(n \geq 4)$ and $H^{3,3}(M,\mathbb{C})$ $(n \geq 6)$ satisfy opposite Cauchy-Schwarz inequality with respect to any Kähler classes in the sense of Definition 1.1. Moreover, the equality case holds if and only if the element is proportional to ω^2 or ω^3 respectively.

Remark 1.5. In [10, Example 1.7], the author described in detail many examples of compact connected Kähler manifolds whose Hodge numbers satisfy $h^{1,1} = 1$ and $h^{1,1} = h^{2,2}$ respectively. These include the complete intersections in complex projective spaces, the complex flag manifolds $G/P_{\max}$ (G is a semisimple complex Lie group and $P_{\max}$ is a maximal parabolic subgroup of G), the one point blow-up of complex projective spaces and so on.

2. Proof of the main result

2.1. **The mixed Hodge-Riemann bilinear relations.** In this subsection we briefly recall the mixed Hodge-Riemann biliner relations established in [2] by Dinh and Nguyên.

As before denote by M an n-dimensional compact connected Kähler manifold. We arbitrarily fix two non-negative integers p, q such that $p, q \leq [\frac{n}{2}]$ and $n-p-q+1$ Kähler classes

$\omega, \omega_1, \ldots, \omega_{n-p-q}$ on M. Put $\Omega := \omega_1 \wedge \cdots \wedge \omega_{n-p-q}$. Define the mixed primitive subspace of $H^{p,q}(M, \mathbb{C})$ *with respect to* ω *and* Ω by

$$P^{p,q}(M; \omega, \Omega) := \{\alpha \in H^{p,q}(M, \mathbb{C}) \mid \alpha \wedge \omega \wedge \Omega = 0\}. \tag{2.1}$$

Define the mixed Hodge-Riemann bilinear form $Q_\Omega(\cdot, \cdot)$ with respect to Ω on $H^{p,q}(M, \mathbb{C})$ by

$$Q_\Omega(\alpha, \beta) := (\sqrt{-1})^{q-p}(-1)^{\frac{(p+q)(p+q+1)}{2}} \int_M \alpha \wedge \bar{\beta} \wedge \Omega, \qquad \alpha, \beta \in H^{p,q}(M, \mathbb{C}). \tag{2.2}$$

Remark 2.1.

(1) Note that this definition of $Q_\Omega(\cdot, \cdot)$ differs from that in (1.3) by a sign when $p = q = 1$.
(2) Clearly when $\omega = \omega_1 = \cdots \omega_{n-p-q}$, $P^{p,q}(M; \omega, \Omega)$ and $Q_\Omega(\cdot, \cdot)$ are nothing but the usual primitive cohomology group and Hodge-Riemann bilinear form with respect to the Kähler class ω.
(3) The symbols $P^{p,q}(M; \omega, \Omega)$ and $Q_\Omega(\cdot, \cdot)$ we use here are simply denoted by $P^{p,q}(M)$ and $Q(\cdot, \cdot)$ respectively in [2]. We use the current symbols to avoid confusion as they stress the dependence on the choices of ω and Ω, whose advantage will be clear in the process of our proof in Theorem 1.3 in the next section.

With the above notation understood, we have the following remarkable result due to Dinh and Nguyên in [2, Theorems A,B,C], which extends the classical Hodge-Riemann biliner relations.

Theorem 2.2 (mixed Hodge-Riemann bilinear relations)**.**

(1) *(mixed Hard Lefschetz theorem) The linear map*

$$\tau : \; H^{p,q}(M, \mathbb{C}) \to H^{n-q,n-p}(M, \mathbb{C})$$

given by

$$\tau(\alpha) := \alpha \wedge \Omega, \qquad \alpha \in H^{p,q}(M, \mathbb{C}) \tag{2.3}$$

is an isomorphism.
(2) *(mixed Lefschetz decomposition) We have the following canonical decomposition:*

$$H^{p,q}(M, \mathbb{C}) = P^{p,q}(M; \omega, \Omega) \oplus \big(\omega \wedge H^{p-1,q-1}(M, \mathbb{C})\big). \tag{2.4}$$

Here $H^{p-1,q-1}(M, \mathbb{C}) := 0$ *if either* $p = 0$ *or* $q = 0$.
(3) *(Positive-definiteness) The mixed Hodge-Riemann bilinear form* $Q_\Omega(\cdot, \cdot)$ *is positive-definite on the mixed primitive subspace* $P^{p,q}(M; \omega, \Omega)$.

Remark 2.3. Note that $P^{p,q}(M; \omega, \Omega)$ depends on ω and Ω while $H^{p,q}(M, \mathbb{C})$ is clearly independent of them. This means, if we fix ω but change $\omega_1, \ldots, \omega_{n-p-q}$, then Ω is also changed respectively and so is $P^{p,q}(M; \omega, \Omega)$. But the mixed Lefschetz decomposition theorem tells us that (2.4) remains true. So the reference Kähler classes $\omega, \omega_1, \ldots, \omega_{n-p-q}$ in context should be clear when we apply (2.4). For unambiguity we shall use the sentence "We apply (2.4) to $\alpha \in H^{p,q}(M, \mathbb{C})$ with respect to the reference Kähler classes ω and $(\omega_1, \ldots, \omega_{n-2p})$" to emphasize it. This notation will play a key role in the proof of (2.6) Lemma 2.4.

2.2. **Proof of Theorem 1.3.** We now apply the mixed Hodge-Riemann bilinear relations to prove our Theorem 1.3.

The following lemma uses the full power of (2.4).

Lemma 2.4.

(1)

$$dim_{\mathbb{C}}P^{p,q}(M;\omega,\Omega)=h^{p,q}-h^{p-1,q-1},\qquad 0\leq p,q\leq[\frac{n}{2}], \tag{2.5}$$

where $h^{p-1,q-1}:=0$ *if either* $p=0$ *or* $q=0$. *This means that the dimension of* $P^{p,q}(M;\omega,\Omega)$ *is independent of* ω *and* Ω *and only depends on the complex structure of* M.

(2) *Let* $\alpha\in H^{p,p}(M,\mathbb{C})$ *with* $1\leq p\leq[\frac{n}{2}]$ *and* $\omega,\omega_1,\dots,\omega_{n-2p}$ *be* $n-2p+1$ *Kähler classes. Put* $\Omega_p:=\omega_1\wedge\cdots\wedge\omega_{n-2p}$. *Then this* α *can be written as follows.*

$$\alpha=\lambda\omega^p+\sum_{i=1}^{p}\alpha_i\wedge\omega^{p-i}, \tag{2.6}$$

where $\lambda\in\mathbb{C}$ *and*

$$\alpha_i\in P^{i,i}(M;\omega,\omega^{2(p-i)}\wedge\Omega_p)\qquad\big(\Leftrightarrow\alpha_i\wedge\omega^{2(p-i)+1}\wedge\Omega_p=0\big). \tag{2.7}$$

(3) $g(\alpha,\omega;\Omega_p)$ *given in Definition 1.1 has the following expression in terms of* α_i:

$$g(\alpha,\omega;\Omega_p)=\big(\int_M\omega^{2p}\wedge\Omega_p\big)\cdot\big(\sum_{i=1}^{p}\int_M\alpha_i\wedge\bar{\alpha}_i\wedge\omega^{2(p-i)}\wedge\Omega_p\big). \tag{2.8}$$

Proof.

(1) The usual Hard Lefschetz theorem ([7, p. 122]) tells us that the map

$$\omega^{n-p-q}\wedge(\cdot):\ H^{p,q}(M,\mathbb{C})\to H^{n-q,n-p}(M,\mathbb{C})$$

is an isomorphism. This means that, for $p+q\leq n-1$, the map

$$\omega\wedge(\cdot):\ H^{p,q}(M,\mathbb{C})\to H^{p+1,q+1}(M,\mathbb{C})$$

is injective and consequently

$$\dim_{\mathbb{C}}\big(\omega\wedge H^{p-1,q-1}(M,\mathbb{C})\big)=h^{p-1,q-1}$$

for $1\leq p,q\leq[\frac{n}{2}]$, which, together with (2.4), leads to (2.5).

(2) The strategy for proving (2.6) is to apply (2.4) repeatedly to yield the desired α_i.

We first apply (2.4) to this α with respect to the reference Kähler classes ω and $(\omega_1,\dots,\omega_{n-2p})$ to yield α_p:

$$\alpha=\alpha_p+\omega\wedge\tilde{\alpha}_{p-1}\qquad\text{with}\qquad\alpha_p\in P^{p,p}(M;\omega,\Omega_p).$$

We continue to apply (2.4) to $\tilde{\alpha}_{p-1}\in H^{p-1,p-1}(M,\mathbb{C})$ with respect to the reference Kähler classes ω and $(\omega,\omega,\omega_1,\dots,\omega_{n-2p})$ to yield α_{p-1}:

$$\tilde{\alpha}_{p-1}=\alpha_{p-1}+\omega\wedge\tilde{\alpha}_{p-2}\qquad\text{with}\qquad\alpha_{p-1}\in P^{p-1,p-1}(M;\omega,\omega^2\wedge\Omega_p).$$

Obviously the next step is to apply (2.4) to $\tilde{\alpha}_{p-2}\in H^{p-2,p-2}(M,\mathbb{C})$ with respect to the reference Kähler classes ω and $(\omega,\omega,\omega,\omega,\omega_1,\dots,\omega_{n-2p})$ to obtain

$$\tilde{\alpha}_{p-2}=\alpha_{p-2}+\omega\wedge\tilde{\alpha}_{p-3}\qquad\text{with}\qquad\alpha_{p-2}\in P^{p-2,p-2}(M;\omega,\omega^4\wedge\Omega_p).$$

Now it is easy to see that repeated use of (2.4) to $\tilde{\alpha}_{p-i}$ determined by $\tilde{\alpha}_{p-i+1}$ with respect to the Kähler classes ω and $(\underbrace{\omega,\dots,\omega}_{2i\text{ copies}},\omega_1,\dots,\omega_{n-2p})$ yields the desired α_{p-i} :

$$\tilde{\alpha}_{p-i}=\alpha_{p-i}+\omega\wedge\tilde{\alpha}_{p-i-1}\qquad\text{with}\qquad\alpha_{p-i}\in P^{p-i,p-i}(M;\omega,\omega^4\wedge\Omega_p).$$

This completes the proof of (2.6).

(3) We now know from (2.6) that

$$\alpha \wedge \omega^p \wedge \Omega_p = \lambda \omega^{2p} \wedge \Omega_p + \sum_{i=1}^{p} \alpha_i \wedge \omega^{2p-i} \wedge \Omega_p \tag{2.9}$$

and

$$\begin{aligned} &\alpha \wedge \bar{\alpha} \wedge \Omega_p \\ =&(\lambda \omega^p + \sum_{i=1}^{p} \alpha_i \wedge \omega^{p-i}) \wedge (\bar{\lambda}\omega^p + \sum_{j=1}^{p} \bar{\alpha}_j \wedge \omega^{p-j}) \wedge \Omega_p \\ =&\big(|\lambda|^2 \omega^{2p} + \lambda \sum_{j=1}^{p} \bar{\alpha}_j \wedge \omega^{2p-j} + \bar{\lambda} \sum_{i=1}^{p} \alpha_i \wedge \omega^{2p-i} + \sum_{i,j=1}^{p} \alpha_i \wedge \bar{\alpha}_j \wedge \omega^{2p-(i+j)}\big) \wedge \Omega_p. \end{aligned} \tag{2.10}$$

Note that

$$\begin{cases} \alpha_i \wedge \omega^{2(p-i)+1} \wedge \Omega_p = 0 \text{ by } (2.7), & 1 \leq i \leq p, \\ 2p - i \geq 2(p-i) + 1, & 1 \leq i \leq p, \\ 2p - (i+j) \geq 2(p - \max\{\text{i,j}\}) + 1, & 1 \leq i \neq j \leq p. \end{cases} \tag{2.11}$$

This means that (2.9) and (2.10) can be simplified via (2.11) as follows.

$$\alpha \wedge \omega^p \wedge \Omega_p = \lambda \omega^{2p} \wedge \Omega_p \tag{2.12}$$

and

$$\alpha \wedge \bar{\alpha} \wedge \Omega_p = |\lambda|^2 \omega^{2p} \wedge \Omega_p + \sum_{i=1}^{p} \alpha_i \wedge \bar{\alpha}_i \wedge \omega^{2(p-i)} \wedge \Omega_p. \tag{2.13}$$

Integrating (2.12) and (2.13) over M deduces that

$$\lambda = \frac{\int_M \alpha \wedge \omega^p \wedge \Omega_p}{\int_M \omega^{2p} \wedge \Omega_p}, \qquad \bar{\lambda} = \frac{\int_M \bar{\alpha} \wedge \omega^p \wedge \Omega_p}{\int_M \omega^{2p} \wedge \Omega_p}, \tag{2.14}$$

and

$$\int_M \alpha \wedge \bar{\alpha} \wedge \Omega_p = \lambda \cdot \bar{\lambda} \cdot \int_M \omega^{2p} \wedge \Omega_p + \sum_{i=1}^{p} \int_M \alpha_i \wedge \bar{\alpha}_i \wedge \omega^{2(p-i)} \wedge \Omega_p. \tag{2.15}$$

(2.8) now follows from substituting the two expressions in (2.14) for the λ and $\bar{\lambda}$ in (2.15).

□

Now we are ready to prove Theorem 1.3, our main result in this article.

Proof. It suffices to prove the first part in Theorem 1.3 as the resulting two cases are similar.

Since $\alpha_i \in P^{i,i}(M; \omega, \omega^{2(p-i)} \wedge \Omega_p)$, the positive-definiteness of the mixed Hodge-Riemann bilinear forms guarantees that $Q_{(\omega^{2(p-i)} \wedge \Omega_p)}(\alpha_i, \alpha_i) \geq 0$ with equality if and only if $\alpha_i = 0$. This, together with the definition of $Q_{(\cdot)}(\cdot, \cdot)$ in (2.2), implies that

$$(-1)^i \int_M \alpha_i \wedge \bar{\alpha}_i \wedge \omega^{2(p-i)} \wedge \Omega_p \geq 0, \qquad 1 \leq i \leq p, \tag{2.16}$$

with the equality holds if and only if $\alpha_i = 0$.

We first show the "if" part of (1) in Theorem 1.3.

The dimension formula (2.5) in Lemma 2.4 and the assumption (1.5) imply that

$$\alpha_{2i+1} = 0, \qquad 0 \leq i \leq [\frac{p+1}{2}] - 1,$$

which, together with (2.8) and (2.16), give us

$$g(\alpha, \omega; \Omega_p) = \big(\int_M \omega^{2p} \wedge \Omega_p\big) \cdot \big(\sum_{\substack{1 \leq i \leq p \\ i \text{ even}}} \int_M \alpha_i \wedge \bar{\alpha}_i \wedge \omega^{2(p-i)} \wedge \Omega_p\big) \geq 0,$$

with equality if and only if all $\alpha_i = 0$ and thus $\alpha \in \mathbb{C}\omega^p$ by the decomposition formula (2.6).

The proof of the "only if" part.

Suppose on the contrary that there exists some $1 \leq i_0 \leq [\frac{p+1}{2}] - 1$ such that $h^{2i_0+1, 2i_0+1} > h^{2i_0, 2i_0}$. Then we can choose a

$$0 \neq \alpha(i_0) \in P^{2i_0+1, 2i_0+1}(M; \omega, \omega^{2\left(p-(2i_0+1)\right)} \wedge \Omega_p)$$

by (2.5) and set

$$\theta := \omega^p + \alpha(i_0) \wedge \omega^{p-(2i_0+1)}.$$

But in this case

$$g(\theta, \omega; \Omega_p) = \big(\int_M \omega^{2p} \wedge \Omega_p\big) \cdot \big(\int_M \alpha(i_0) \wedge \bar{\alpha(i_0)} \wedge \omega^{2\left(p-(2i_0+1)\right)} \wedge \Omega_p\big) < 0$$

and thus this θ does not satisfy Cauchy-Schwarz inequality with respect to the Kähler classes ω and $(\omega_1, \ldots, \omega_{n-2p})$ in the sense of Definition 1.1, which contradicts to the assumption. This gives the desired proof. □

The corollary below follows from the process of the above proof.

Corollary 2.5. Suppose $\alpha \in H^{p,p}(M, \mathbb{C})$ with $1 \leq p \leq [\frac{n}{2}]$. Then this α satisfies Cauchy-Schwarz (resp. opposite Cauchy-Schwarz) inequality with respect to Kähler classes $\omega, \omega_1, \ldots, \omega_{n-2p}$ if those α_i with i odd (resp. even) determined by the decomposition formula (2.8) all vanish.

3. A proportionality problem

Let $\mathcal{K} \in H^{1,1}(M, \mathbb{R})$ be the Kähler cone of M, which consists of all the Kähler classes of M. Recall that $c \in H^{1,1}(M, \mathbb{R})$ is called a *nef* class if $c \in \bar{\mathcal{K}}$, the closure of the Kähler cone. So nef classes can be approximated by Kähler classes. This means Theorem 1.3 has the following corollary.

Corollary 3.1. Suppose M is an n-dimensional compact connected Kähler manifold.

(1) Given $1 \leq p \leq [\frac{n}{2}]$, if the Hodge numbers of M satisfy

$$h^{2i,2i} = h^{2i+1,2i+1}, \qquad 0 \leq i \leq [\frac{p+1}{2}] - 1,$$

then for any $\alpha \in H^{p,p}(M, \mathbb{C})$ and any nef classes $c, c_1, \ldots, c_{n-2p}$ we have

$$\big(\int_M \alpha \wedge \bar{\alpha} \wedge C_p\big) \cdot \big(\int_M c^{2p} \wedge C_p\big) \geq \big(\int_M \alpha \wedge c^p \wedge C_p\big) \cdot \big(\int_M \bar{\alpha} \wedge c^p \wedge C_p\big), \tag{3.1}$$

where $C_p = c_1 \wedge \cdots \wedge c_{n-2p}$.

(2) For any $\alpha \in H^{1,1}(M,\mathbb{C})$ and any nef classes $c, c_1, \ldots, c_{n-2}$ we have

$$(3.2)\qquad \Big(\int_M \alpha\wedge\bar{\alpha}\wedge C\Big)\cdot\Big(\int_M c^{2p}\wedge C\Big)\leq\Big(\int_M \alpha\wedge c^p\wedge C\Big)\cdot\Big(\int_M \bar{\alpha}\wedge c^p\wedge C\Big),$$

where $C = c_1\wedge\cdots\wedge c_{n-2}$.

(3) Given $2 \leq p \leq [\frac{n}{2}]$, if the Hodge numbers of M satisfy

$$h^{2i-1,2i-1} = h^{2i,2i}, \qquad 1 \leq i \leq [\frac{p}{2}],$$

then for any $\alpha \in H^{p,p}(M,\mathbb{C})$ and any nef classes $c, c_1, \ldots, c_{n-2p}$ we have

$$(3.3)\qquad \Big(\int_M \alpha\wedge\bar{\alpha}\wedge C_p\Big)\cdot\Big(\int_M c^{2p}\wedge C_p\Big)\leq\Big(\int_M \alpha\wedge c^p\wedge C_p\Big)\cdot\Big(\int_M \bar{\alpha}\wedge c^p\wedge C_p\Big),$$

where $C_p = c_1\wedge\cdots\wedge c_{n-2p}$.

However, unlike Theorem 1.3 for Kähler classes, we can *not* conclude directly in this case that the equalities in (3.1), (3.2) and (3.3) hold if and only if α and the nef class c are proportional. So a natural question is to characterize the equalities in (3.1)-(3.3), a very special case of which has been proposed by Teissier in [12] as a further question related to his inequality (1.1) and we shall briefly reivew it in what follows.

(1.1) or (3.2) gives us that, for two *nef* divisors D_1 and D_2 on an algebraic manifold and for $1 \leq k \leq n-1$, we have

$$(3.4)\qquad \big([D_1^k D_2^{n-k}]\big)^2 \geq [D_1^{k-1}D_2^{n-k+1}]\cdot[D_1^{k+1}D_2^{n-k-1}].$$

Teissier considered in [12] that how to characterize the equality case in (3.4) for nef and *big* divisors D_1 and D_2 (recall that a nef divisor D is called *big* if moreover $[D^n] > 0$). This problem was solved in [3, Theorem D] by Boucksom, Favre and Jonsson, whose result asserts that for two nef and big divisors D_1 and D_2 the equality in (3.4) holds if and only if D_1 and D_2 are numerically proportional. Very recently Fu and Xiao obtained the same type result in the context of Kähler manifolds ([5, Theorem 2.1]), some of whose ideas are based on their previous work in [4].

Remark 3.2. The expression used in [3, Theorem D, (2)] is slightly different from our (3.4) but they are indeed equivalent (see, for instance, the equivalent statements in ([5, Theorem 2.1]).

With the Cauchy-Schwarz-type inequalities in its full generality in Corollary 3.1 in hand, we can now end our article by posing the following problem, whose solution is obviously beyond the content of this note.

Question 3.3. How to characterize the three inequality cases in (3.1), (3.2) and (3.3)? Clearly α being proportional to c^p is a sufficient condition. Is this also a necessary condition? Or weakly how to establish such a necessary condition by imposing more constraints on the element α, the nef classes $c, c_1, \ldots, c_{n-2p}$ and/or the underlying Kähler manifold M?

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